Math. Appl. **2** (2013), 35–42

SOME WOLSTENHOLME TYPE CONGRUENCES

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Abstract. In this paper we give an extension and another proof of the following Wolstenholme's type curious congruence established in 2008 by J. Zhao. Let s and l be two positive integers and let p be a prime such that $p \geq ls + 3$. Then

$$H(\{s\}^l; p-1) \equiv S(\{s\}^l; p-1) \equiv \begin{cases} -\dfrac{s(ls+1)p^2}{2(ls+2)} B_{p-ls-2} & (\bmod p^3) \quad \text{if } 2 \nmid ls \\ (-1)^{l-1} \dfrac{sp}{ls+1} B_{p-ls-1} & (\bmod p^2) \quad \text{if } 2 \mid ls. \end{cases}$$

As an application, for given prime $p \geq 5$, we obtain explicit formulae for the sum $\sum_{1 \leq k_1 < \cdots < k_l \leq p-1} 1/(k_1 \cdots k_l) \pmod{p^3}$ if $k \in \{1, 3, \ldots, p-2\}$, and for the sum $\sum_{1 \leq k_1 < \cdots < k_l \leq p-1} 1/(k_1 \cdots k_l) \pmod{p^2}$ if $k \in \{2, 4, \ldots, p-3\}$.

1. Introduction and basic results

Our investigations are motivated by some recent results to multiple harmonic sums obtained by J. Zhao [12], Zhou and Cai [13]. These results are in fact, variations and generalizations of Wolstenholme's theorem. For more information on extensions and generalizations of Wolstenholme's theorem, see [5], [9], [10] and [11].

Throughout this paper we use the following definitions and notations.

For $n \in \mathbb{N}$, $l \in \mathbb{N}$ and $\mathbf{s} := (s_1, \ldots, s_l) \in \mathbb{N}^l$, define the finite *harmonic sum*

$$H(\mathbf{s}; n) := H(s_1, \ldots, s_l; n) = \sum_{1 \leq k_1 < \cdots < k_l \leq n} \frac{1}{k_1^{s_1} \cdots k_l^{s_l}}.$$

By convention we set $H(\mathbf{s}; r) = 0$ for $r = 0, \ldots, l-1$. Further, we define the sum

$$S(\mathbf{s}; n) := S(s_1, \ldots, s_l; n) = \sum_{1 \leq k_1 \leq \cdots \leq k_l \leq n} \frac{1}{k_1^{s_1} \cdots k_l^{s_l}}.$$

If $s_1 = \ldots = s_l = s$ then $H(\mathbf{s}; n)$ is a *homogeneous harmonic sum*. In this case, we shall denote such a sum by $H(\{s\}^l; n)$, and hence

$$H(\{s\}^l; n) = \sum_{1 \leq k_1 < \cdots < k_l \leq n} \frac{1}{(k_1 \cdots k_l)^s}.$$

MSC (2010): primary 11B65; secondary 11A07, 05A10.

Keywords: multiple harmonic sum, Wolstenholme's theorem, Bernoulli numbers, Stirling numbers.

In particular, we write $H(s;n)$ instead of $H(\{s\}^1;n)$, that is,

$$H(s;n)=\sum_{1\le k\le n}\frac{1}{k^s}.$$

Analogously, we define $S(\{s\}^l;n)$ and $S(s;n)$ related to the sums $S(\mathbf{s};n)$.

Recall that *Stirling numbers* $St(n,j)$ *of the first kind* are defined by the expansion

$$x(x+1)(x+2)\cdots(x+n-1)=\sum_{j=1}^{n}St(n,j)x^j.$$

It is easy to see that for all $j=1,\dots,n-1$,

$$St(n,j)=\sum_{1\le k_1<\cdots<k_{n-j}\le n-1}k_1\cdots k_{n-j},$$

and that

$$St(n,j+1)=(n-1)!\cdot H(\{1\}^j;n-1).$$

For example, $St(n,n)=1$, $St(n,n-1)=n(n-1)/2$, and $St(n,1)=(n-1)!$.

Further, for any nonnegative integers j and $n\ge 1$, jth *power-sum symmetric function* is defined as

$$P(n,j)=\sum_{k=1}^{n}k^j.$$

By convention we set $P(0,j)=0$ for all $j\ge 0$.

By *Wolstenholme's theorem* (see, e.g., [6, p. 89]), if p is a prime greater than 3, then the numerator of the fraction

$$H(1,p-1)=1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{p-1}$$

written in reduced form, is divisible by p^2.

Denote by $\mathfrak{p}$ the parity of m which is 1 if m is odd and 2 if m is even. Bayat proved

Theorem A ([1, Theorem 3]; also see Remark 2.3 in [12])**.** *For any positive integer s and a prime $p\ge s+3$ we have*

$$H(s;p-1)\equiv 0 \pmod{p^{\mathfrak{p}(s+1)}}.$$

Zhao in [12, p. 74] reported that one can find on the Internet the following generalization of Wolstenholme's theorem by Bruck [2], although no proof is given here.

Theorem B ([12, Theorem 1.2])**.** *For any prime number $p\ge 5$ and positive integers $l=1,\dots,p-3$, we have*

$$St(p,l+1)\equiv 0 \pmod{p^{\mathfrak{p}(l+1)}},$$

and

$$H(\{1\}^l;p-1)\equiv 0 \pmod{p^{\mathfrak{p}(l+1)}}.$$

The *Bernoulli numbers* B_k are defined by the generating function

$$\sum_{k=0}^{\infty} B_k \frac{x^k}{k!} = \frac{x}{e^x - 1}\,.$$

It is easy to find the values $B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$, and $B_n = 0$ for odd $n \geq 3$. Furthermore, $(-1)^{n-1}B_{2n} > 0$ for all $n \geq 1$. These and many other properties can be found, for instance, in [7] and [3].

Recently, Zhao in [12] proved the following generalization of Theorem B to homogeneous multiple harmonic sums.

Theorem C ([12, Theorem 2.14]; cf. [12, Theorem 1.6]). *Let s and l be two positive integers. Let p be a prime such that $p \geq ls + 3$. Then*

$$H(\{s\}^l; p-1) \equiv S(\{s\}^l; p-1) \equiv \begin{cases} -\dfrac{s(ls+1)p^2}{2(ls+2)} B_{p-ls-2} \pmod{p^3} & \textit{if } 2 \nmid ls \\ (-1)^{l-1} \dfrac{sp}{ls+1} B_{p-ls-1} \pmod{p^2} & \textit{if } 2 \mid ls. \end{cases}$$

As an application, Zhao obtained the following result.

Theorem D ([12, Proposition 2.15]; also cf. [12, Theorem 1.5]). *Let s and l be two positive integers. Let p be a prime such that $p \geq l + 2$ and $p - 1$ divides none of ks and $ks + 1$ for $k = 1, \ldots, l$. Then*

$$H(\{s\}^l; p-1) \equiv S(\{s\}^l; p-1) \equiv 0 \pmod{p^{\mathfrak{p}(ls-1)}}.$$

In particular, if $p \geq ls+3$, then the above is always true, and so $p \mid H(\{s\}^l; p-1)$.

As noticed in [12, p. 85, Proof of Theorem 2.14], the above congruence for $H(\{s\}^l; p-1)$ follows immediately from [13, Lemma 2], while the above congruence for $S(\{s\}^l; p-1)$ then follows from the equality (2.11) in [12] and induction on l.

In Section 2 we give another proof of the above congruence for $H(\{s\}^l; p-1)$. Our result is as follows.

Theorem 1.1. *Let s and l be two positive integers. Let p be a prime such that $p \geq ls + 3$. Then*

$$H(\{s\}^l; p-1) \equiv (-1)^{l-1} \frac{H(ls; p-1)}{l} \equiv \begin{cases} -\dfrac{s(ls+1)p^2}{2(ls+2)} B_{p-ls-2} \pmod{p^3} \ \textit{if } 2 \nmid ls \\ (-1)^{l-1} \dfrac{sp}{ls+1} B_{p-ls-1} \pmod{p^2} \ \textit{if } 2 \mid ls. \end{cases}$$

Taking $s = 1$ in Theorem 1.1, we obtain the following result.

Corollary 1.2. *Let p be a prime greater than 3, and let l be a positive integer such that $l \leq p - 3$. Then*

$$H(\{1\}^l; p-1) := \sum_{1 \leq k_1 < \cdots < k_l \leq p-1} \frac{1}{k_1 \cdots k_l} \equiv \begin{cases} -\dfrac{(l+1)p^2}{2(l+2)} B_{p-l-2} \pmod{p^3} \ \textit{if } 2 \nmid l \\ -\dfrac{p}{l+1} B_{p-l-1} \pmod{p^2} \ \textit{if } 2 \mid l. \end{cases}$$

Corollary 1.3. *Let p be a prime greater than 3, and let l be a positive integer such that $2 \le l \le p-2$. Then*

$$St(p,l) \equiv \begin{cases} \dfrac{lp^2}{2(l+1)} B_{p-1-l} & (\bmod\, p^3) \text{ } if \text{ } 2 \mid l \\ \dfrac{p}{l} B_{p-l} & (\bmod\, p^2) \text{ } if \text{ } 2 \nmid l. \end{cases}$$

Proof. Since by Wilson theorem, $(p-1)! \equiv -1 \ (\bmod\, p)$, we get for all $l = 2,\dots,p-2$

$$St(p,l) = (p-1)! \cdot H(\{1\}^{l-1}; p-1) \equiv -H(\{1\}^{l-1}; p-1) \pmod{p}.$$

Therefore, both congruences follow immediately from the congruences given in Corollary 1.2. □

Remark 1.4. Observe that Corollary 1.2 does not contain a related congruence for the Stirling number $St(p,1) = (p-1)!$. In 1900 Glaisher [4] showed that $(p-1)! \equiv pB_{p-1} - p \ (\bmod\, p^2)$.

In order to prove Theorem 1.1, we use Theorem A in the proof of Lemma 2.2 given in Section 2. Note that Theorem A is an immediate consequence of a classical result of E. Lehmer [8] given by Lemma 2.7. On the other hand, Theorem B is an immediate consequence of Corollaries 1.2 and 1.3 and the fact that by Lemma 2.5, the denominator of the Bernoulli number B_l, written in reduced form, is not divisible by p for each integer l such that $0 \le l \le p-3$. Further, the congruence for $H(\{s\}^l; p-1)$ in Theorem C is given by Theorem 1.1. Finally, note that (see Remark 2.3) the proof of Theorem D is the same as that of Lemma 2.2.

2. Proof of Theorem 1.1

For the proof of Theorem 1.1, we will need some auxiliary results.

Lemma 2.1. *Let n, s and l be positive integers such that $l \ge 2$ and $n \ge ls$. Then*

$$\sum_{j=1}^{l-1} (-1)^{j-1} H(js;n) \cdot H(\{s\}^{l-j};n) = lH(\{s\}^l;n) + (-1)^l H(ls;n). \tag{2.1}$$

Proof. First note that (2.1) is trivially satisfied for $l=2$. For simplicity, here we write $H(\{\mathbf{s}\}^l)$ instead of $H(\{\mathbf{s}\}^l;n)$, and denote

$$\sigma(j) = \sum_{i=0}^{l-j} H(\underbrace{s,\dots,s}_{i}, js, \underbrace{s,\dots,s}_{l-j}; n), \quad j = 1,\dots l,$$

whence we see that $\sigma(l) = H(ls; n)$. Now if $l \geq 3$, then for all j with $2 \leq j \leq l-1$, we have

$$\begin{aligned}
H(js)\cdot H(\{s\}^{l-j}) &= \left(\sum_{1\leq k\leq n}\frac{1}{k^{js}}\right)\left(\sum_{1\leq k_1<\cdots<k_{l-j}\leq n}\frac{1}{(k_1\cdots k_{l-j})^s}\right)\\
&= \sum_{k\notin\{k_1,\dots,k_{l-j}\}} + \sum_{k\in\{k_1,\dots,k_{l-j}\}}\\
&= \sum_{i=0}^{l-j} H(\underbrace{s,\dots,s}_{i}, js, \underbrace{s,\dots,s}_{l-j-i})\\
&\quad + \sum_{i=0}^{l-j-1} H(\underbrace{s,\dots,s}_{i}, (j+1)s, \underbrace{s,\dots,s}_{l-j-1-i})\\
&= \sigma(j) + \sigma(j+1).
\end{aligned}$$

Furthermore, for $j = 1$, we have

$$\begin{aligned}
H(s)\cdot H(\{s\}^{l-1}) &= \left(\sum_{1\leq k\leq n}\frac{1}{k^{s}}\right)\left(\sum_{1\leq k_1<\cdots<k_{l-1}\leq n}\frac{1}{(k_1\cdots k_{l-1})^s}\right)\\
&= \sum_{k\notin\{k_1,\dots,k_{l-1}\}} + \sum_{k\in\{k_1,\dots,k_{l-1}\}}\\
&= \sum_{i=0}^{l-1} H(\underbrace{s,\dots,s}_{i}, s, \underbrace{s,\dots,s}_{l-1-i}) + \sum_{i=0}^{l-1} H(\underbrace{s,\dots,s}_{i}, 2s, \underbrace{s,\dots,s}_{l-2-i})\\
&= lH(\{s\}^l) + \sigma(2).
\end{aligned}$$

The above two equalities imply

$$\begin{aligned}
\sum_{j=1}^{l-1}(-1)^{j-1}H(js)\cdot H(\{s\}^{l-j}) &= lH(\{s\}^l) + \sigma(2) + \sum_{j=2}^{l-1}(-1)^{j-1}(\sigma(j)+\sigma(j+1))\\
&= lH(\{s\}^l) + \sigma(2) + \sum_{j=2}^{l-1}(-1)^{j-1}(\sigma(j)+\sigma(j+1))\\
&= lH(\{s\}^l) + \sigma(2) - \sigma(2) + (-1)^{l-1}\sigma(l)\\
&= lH(\{s\}^l) + (-1)^l H(ls; n),
\end{aligned}$$

as desired. □

The following lemma is an extension of the congruence for harmonic sums $H(\{1\}^l; p-1)$ given by Theorem B. This is in fact the congruence for $H(\{s\}^l; p-1)$ from Theorem D when $p \geq ls+3$.

Lemma 2.2 (cf. Theorem C)**.** *Let s and l be two positive integers, and let p be a prime such that $p \geq ls+3$. Then $p^2 \mid H(\{s\}^l; p-1)$ if ls is odd, and $p \mid H(\{s\}^l; p-1)$ if ls is even.*

Proof. Putting $n = p-1$ in (2.1) of Lemma 2.1, we obtain

$$\sum_{j=1}^{l-1}(-1)^{j-1}H(js;p-1)\cdot H(\{s\}^{l-j};p-1) + (-1)^{l-1}H(ls;p-1) = lH(\{s\}^l;p-1). \quad (2.2)$$

We proceed by induction on the sum $\sigma := l+s \geq 2$. If $\sigma = 2$ then $l = s = 1$, and $p^2 \mid H(1;p-1)$ by Wolstenholme's theorem. Now suppose that the assertion is true for some σ with $\sigma \geq 2$. This means that $p^2 \mid H(\{s'\}^{l'};p-1)$ whenever l' and s' are both odd such that $l'+s' \leq \sigma$ and $p \geq l's'+3$, and that $p \mid H(\{s'\}^{l'};p-1)$ whenever $l's'$ is even such that $l'+s' \leq \sigma$ and $p \geq l's'+3$ In order to prove the assertion for all pairs l and s with $l+s = \sigma$ and $p \geq ls+3$, we consider the following two cases.

`Case 1.` ls is odd; that is both integers l and s are odd. Then, for odd j with $1 \leq j \leq l-1$, we have $s+(l-j) = \sigma - j < \sigma$ and $s(l-j)$ is even. Therefore, by the inductive hypothesis, $p \mid H(\{s\}^{l-j}$. Furthermore, for such a j, by Theorem A, $p^2 \mid H(js;p-1)$.

Similarly, if j is even with $1 \leq j \leq l-1$, we also have $s+(l-j) = \sigma-j < \sigma$, and $s(l-j)$ is odd. Thus, by the inductive hypothesis, $p^2 \mid H(\{s\}^{l-j}$. Furthermore, for such a j, by Theorem A, $p \mid H(js;p-1)$.

Hence, in both cases it follows that $p^3 \mid H(js;p-1)\cdot H(\{s\}^{l-j};p-1)$. This together with the fact that, by Theorem A, $p^3 \mid H(ls;p-1)$, implies that the sum on the right hand side of (2.2) is divisible by p^3. Therefore, $p^3 \mid lH(\{s\}^l;p-1)$, whence, because $l \leq ls \leq p-3$, it follows that $p^3 \mid H(\{s\}^l;p-1)$. This concludes the inductive proof when ls is odd.

`Case 2.` ls is even. Then in the same way as in the first case, we obtain by induction on the sum $l+s$ that $p^2 \mid H(\{s\}^l;p-1)$.

This completes the inductive proof. $\square$

Remark 2.3. Observe that the above proof holds if we replace the condition $p \geq ls+3$ of the Lemma by the following conditions of Theorem D: $p \geq l+2$ and $p-1$ divides none of ks and $ks+1$ for $k = 1,\ldots,l$. In other words, Theorem D can be proved in the same manner as Lemma 2.2.

We are now ready to state the following

Proposition 2.4. *Let s and l be two positive integers, and let $t = \max\{1, l-1\}$. Let p be a prime such that $p \geq ts+3$. Then*

$$lH(\{s\}^l;p-1) \equiv (-1)^{l-1}H(ls;p-1) \begin{cases} \pmod{p^3} & if\ 2 \nmid ls \\ \pmod{p^2} & if\ 2 \mid ls. \end{cases}$$

Proof. If $l = 1$ and $p \geq s+3$, then the above congruences reduce to the congruence for $H(s;p-1)$ given in Theorem A. If $l \geq 2$ and $p \geq (l-1)s+3$, by (2.2) of the proof of Lemma 2.2, we have

$$\sum_{j=1}^{l-1}(-1)^{j-1}H(js;p-1)\cdot H(\{s\}^{l-j};p-1) = lH(\{s\}^l;p-1) - (-1)^l H(ls;p-1).$$

From the proof of Lemma 2.2, we see that each term of the sum on the left hand side of the above identity is divisible by p^3 if ls is odd, and by by p^2 if ls is even. Clearly, this fact implies both congruences from our Proposition. $\square$

Lemma 2.5. *Let $p \geq 3$ be a prime, and let s be any even integer such that $0 \leq s \leq p-3$. Then the denominator of the Bernoulli number B_s, written in reduced form, is not divisible by p.*

Proof. If $p = 3$, then $s = 0$, that is, $B_0 = 1$. Suppose now that $p \geq 5$. It is well known (see [7]) that Bernoulli numbers can be defined recursively as

$$B_s = -\frac{1}{s+1}\sum_{i=0}^{s-1}\binom{s+1}{i}B_i.$$

Now, by induction on even s with $0 \leq s \leq p-3$, the above equality immediately implies that the denominator of B_s, written in reduced form, is not divisible by p. $\square$

Remark 2.6. The above lemma is an immediate consequence of the von Staudt-Clausen theorem, which asserts that $B_{2m} + \sum_{p-1|2m} 1/p$ is an integer for all $m \in \mathbb{N}$, where the summation is over all primes p such that $p-1 \mid 2m$ (see, for example, [7, p. 233, Theorem 3]). If $B_{2m} = N_{2m}/D_{2m}$ with $\gcd(N_{2m}, D_{2m}) = 1$, then, by this result, it follows that the denominator D_{2m} of B_{2m} is given by

$$D_{2m} = \prod_{\substack{p\ prime\\ p-1|2m}} p,$$

whence Lemma 2.5 follows.

The following result is closely related to congruences of Glaisher [4], as quoted and proved by E. Lehmer (two congruences after (16) in [8]).

Lemma 2.7 ([8]; also cf. [12, Theorem 2.8])**.** *Let $p \geq 5$ be a prime, and let m be any integer such that $1 \leq m \leq (p-3)/2$. Then*

$$H(2m-1; p-1) := \sum_{k=1}^{p-1}\frac{1}{k^{2m-1}} \equiv \frac{m(1-2m)p^2}{2m+1}B_{p-1-2m} \pmod{p^3}, \tag{2.3}$$

and

$$H(2m; p-1) := \sum_{k=1}^{p-1}\frac{1}{k^{2m}} \equiv \frac{2mp}{2m+1}B_{p-1-2m} \pmod{p^2}. \tag{2.4}$$

Proof. Consider the case when ls is odd, that is, when both l and s are odd. By the first congruence of Proposition 2.4 and the congruence (2.3) of the above lemma with $ls = 2m-1$, we obtain

$$\begin{aligned} H(\{s\}^l; p-1) &\equiv \frac{H(ls;p-1)}{l} && \pmod{p^3}\\ &\equiv -\frac{1}{l}\frac{sl(ls+1)p^2}{2(ls+2)}B_{p-ls-2} && \pmod{p^3}\\ &= -\frac{s(ls+1)p^2}{2(ls+2)}B_{p-ls-2} && \pmod{p^3}. \end{aligned}$$

Similarly, using the second congruence of Proposition 2.4 and the congruence (2.4) of the above lemma with $ls = 2m$, we obtain the congruence of Theorem 1.1 for even ls. □

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